PICARD GROUPS OF PUNCTURED SPECTRA OF DIMENSION THREE LOCAL HYPERSURFACES ARE TORSION-FREE

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ABSTRACT. Let (R, m) be a Noetherian local ring and $U_R = \operatorname{Spec}(R) - \{m\}$ be the punctured spectrum of R. Gabber conjectured that if R is a complete intersection of dimension 3, then the abelian group $\operatorname{Pic}(U_R)$ is torsion-free. In this note we prove Gabber's statement for the hypersurface case. We also point out certain connections between Gabber's Conjecture, Van den Bergh's notion of non-commutative crepant resolutions and some well-studied questions in homological algebra over local rings.

1. Introduction

Let (R, m) be a local ring (always Noetherian in this note). Let $U_R = \operatorname{Spec}(R) - \{m\}$ be the punctured spectrum of R. In [12] Gabber made the following:

Conjecture 1.1. *Let R be a local complete intersection of dimension 3. Then $\operatorname{Pic}(U_R)$ is torsion-free.*

The above Conjecture is equivalent to the statement that the local flat cohomology group $H^2_{\{m\}}(\operatorname{Spec}(R), \mu_n) = 0$ when R is a local complete intersection of dimension 3, and they are both implied by (for more details, see [12]):

Conjecture 1.2. *Let R be a strictly henselian local complete intersection of dimension at least 4. Then the cohomological Brauer group of U_R vanishes: $Br(U_R) = 0$.*

Conjecture 1.1 is known when R contains a field; the characteristic 0 case follows from Grothendieck's techniques on local Leftschetz theorems (cf. [3, 25]), and the positive characteristic case can be found in [9] (it is probably known to experts, though we can not find an exact reference. It was claimed in [12] that Conjecture 1.2 is known in positive characteristic). We also note that when U_R is replaced by a smooth projective complete intersection the analogous result on the Picard group is contained in [8, Theorem 1.8]. In any case, the main difficulty is when R is of mixed characteristic.

In this paper we give a short and relatively self-contained proof of Gabber's Conjecture 1.1 for the case of hypersurfaces, that is, if $\hat{R} \cong T/(f)$ where T is a complete regular local ring. In fact, in this situation we shall prove a stronger result which is a pure commutative algebra statement. To state such result let us recall a useful notion. For a Noetherian ring R one can define a map $c_1 : G(R) \to \operatorname{CH}^1(R)$ from the Grothendieck group of finitely generated modules over R to the height

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one component of the Chow group of $\mathrm{Spec}(R)$ (see Subsection 2.2 for more details). Given an R-module M, we shall abuse notation a bit and call $c_1([M])$ the first local Chern class of M. Then our main result says:

Theorem 1.3. *Let R be local hypersurface of dimension 3. Let N be a finitely generated reflexive R-module which is locally free on U_R. Furthermore, assume that the first local Chern class of N is torsion in $\mathrm{CH}^1(R)$. Then $\mathrm{Hom}_R(N,N)$ is a maximal Cohen-Macaulay R-module if and only if N is free.*

It is not hard to see that the above Theorem implies Conjecture 1.1 in the hypersurfaces case, by taking N to be the R-module generated by the sections of a torsion element in $\mathrm{Pic}(U_R)$; see section 2 and the proof of 3.5 for more details.

This project actually arises from our attempt to understand a striking definition by Van den Bergh of non-commutative crepant resolutions of a Gorenstein local ring R. To explain the connection we recall:

Definition 1.4. *(Van den Bergh, [26]) Suppose that there exists a reflexive module N satisfying:*

1. *$A = \mathrm{Hom}_R(N,N)$ is a maximal Cohen-Macaulay R-module.*
2. *A has finite global dimension equal to $d = \dim R$.*

Then A is called a non-commutative crepant resolution (henceforth NCCR) of R.

In [7] we proved that non-commutative crepant resolutions can not exist when R is a dimension 3, equicharacteristic or unramified hypersurface with isolated singularity and torsion class group. Theorem 1.3 implies:

Corollary 1.5. *Let R be a dimension 3 hypersurface which has isolated singularity and torsion class group (which in this case is equivalent to R being an unique factorization domain by our main results). Then R has no non-commutative crepant resolution in the sense of Van den Bergh.*

We now briefly describe the organization of the paper. Section 2 deals with preliminary materials. In Section 3 we give the proofs of the main results announced above as well as some other interesting applications. Finally, in Section 4 we raise some open questions relevant to our approach to Gabber's conjecture.

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2. Notations and preliminary results

Throughout the note R will be a Noetherian local ring. Recall that a maximal Cohen-Macaulay R-module M is a finitely generated module satisfying $\operatorname{depth} M = \dim R$.

Let $\mathrm{mod}(R)$ and $\mathrm{MCM}(R)$ be the category of finitely generated and finitely generated maximal Cohen-Macaulay R-modules, respectively. Suppose X is a Noetherian scheme. Let $\mathfrak{Coh}(X)$ denote the category of coherent sheaves on X and $\mathfrak{Vect}(X)$ the subcategory of vector bundles on X. By $G(X), \mathrm{Pic}(X), \mathrm{CH}^i(X), \mathrm{Cl}(X)$ we shall denote the Grothendieck group of coherent sheaves on X, the Picard group of invertible sheaves on X, the Chow group of codimension i irreducible, closed subschemes of X, and the class group of X, respectively. When $X = \operatorname{Spec} R$ we shall write $G(R), \mathrm{Pic}(R), \mathrm{CH}^i(R), \mathrm{Cl}(R)$. Let $\overline{G}(R) := G(R)/\mathbb{Z}[R]$ be the reduced Grothendieck group and $\overline{G}(R)_{\mathbb{Q}} := \overline{G}(R) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the reduced Grothendieck group of R with rational coefficients.

2.1. **Vector bundles on U_R and modules over R.** Let Γ_X be the section functor on X. We have the following:

Proposition 2.1. *(Horrocks* [17, section 1]*) Let R be a Noetherian local ring such that* $\operatorname{depth} R \geq 2$. *Let $X = U_R$. Then Γ_X induces an equivalence of category between $\mathfrak{Vect}(X)$ and the subcategory of* $\operatorname{mod}(R)$ *consisting of finitely generated modules M which is locally free on non-maximal primes with* $\operatorname{depth} M \geq 2$ *(Note that the condition* $\operatorname{depth} R \geq 2$ *also ensures that X is connected).*

In particular, let $\mathcal{E}$ represent an element in $\operatorname{Pic}(X)$ and let $I = \Gamma_X(\mathcal{E})$. We know that I is a reflexive ideal in R which is locally free of rank 1 on X. Furthermore $\operatorname{Hom}_R(I, I) \cong \Gamma_X(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \cong \Gamma_X(\mathcal{O}_X) = R$.

2.2. **Some maps between Chow, Picard, and Grothendieck groups.** In this Subsection we assume that R is a local ring such that $\operatorname{depth} R \geq 2$. For $i = 0, 1$ there are maps $c_i : G(R) \to \operatorname{CH}^i(R)$. These maps admit a very elementary definition as follows: suppose M is an R-module. Pick any prime filtration $\mathcal{F}$ of M. Then one can take $c_i([M]) = \sum [R/p]$, where p runs over all prime ideals such that R/p appears in $\mathcal{F}$ and $\operatorname{height}(p) = i$, note that a prime can occur multiple times in the sum (for a proof that this is well-defined see the main Theorem of [4]). When R is a normal, algebra essentially of finite type over a field and N is locally free (i.e. a vector bundle) on U_R, c_1 agrees with the first Chern class of N, as defined in [11, Chapter 3], but we shall not need that fact.

One has the following diagram of maps of abelian groups:

$$\begin{tikzcd} & \operatorname{Pic}(U_R) \arrow[d, "p"] \\ G(R) \arrow[r, "c_1"] & \operatorname{CH}^1(R) \end{tikzcd}$$

Here p is induced by the well-known map between Cartier and Weil divisors (see Chapter 2 of [11]).

Note that we do not indicate any map between $\operatorname{Pic}(U_R)$ and $G(R)$. However, the diagram "commutes" in a weak sense: if $\mathcal{E}$ represents an element in $\operatorname{Pic}(X)$ and $I = \Gamma_X(\mathcal{E})$ then $p([\mathcal{E}]) = c_1([I])$ in $\operatorname{CH}^1(R)$.

Obviously, $c_1([R]) = 0$, so c_1 induces a map $q : \overline{G}(R) \to \operatorname{CH}^1(R)$. In particular, if M is a module such that $[M] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$ then $c_1([M])$ is torsion in $\operatorname{CH}^1(R)$.

2.3. **Maximal Cohen-Macaulay approximations.** The reference for this Subsection is the paper [1]. Suppose that R is Cohen-Macaulay and a homomorphic image of a Gorenstein ring. For any R-module N there exists a short exact sequence:

$$0 \to W \to M \to N \to 0 \tag{2.1}$$

such that $M \in \operatorname{MCM}(R)$ and W has finite injective dimension. Note that if R is Gorenstein, then $\operatorname{pd}_R W < \infty$. Also, if R is Gorenstein and $\operatorname{depth} N \geq \dim R - 1$, then by counting depth and the Auslander-Buchsbaum formula W must be free.

2.4. **Hochster's theta function.** Let R be a local hypersurface, so $\hat{R} = T/(f)$ where T is a regular local ring. Suppose that M is an R module such that $\operatorname{pd}_{R_p} M_p < \infty$ for any $p \in U_R$. Then for any R-module N, $\ell(\operatorname{Tor}_i^R(M, N)) < \infty$

for $i \gg 0$; here $\ell(-)$ denotes length. The function $\theta^R(M,N)$ was introduced by Hochster ([16]) to be:

$$\theta^R(M,N) = \ell(\mathrm{Tor}^R_{2e+2}(M,N)) - \ell(\mathrm{Tor}^R_{2e+1}(M,N))$$

where e is any integer such that $2e \geq \dim R$. It is well known (see [10]) that the sequence of modules $\{\mathrm{Tor}^R_i(M,N)\}$ is periodic of period 2 for $i > \operatorname{depth} R - \operatorname{depth} M$, so this function is well-defined. The theta function satisfies the following properties:

Proposition 2.2. *(Hochster,* [16]*)*

(1) *If $M \otimes_R N$ has finite length, then:*
$$\theta^R(M,N) = \chi^T(M,N) := \sum_{i \geq 0} (-1)^i \ell(\mathrm{Tor}^T_i(M,N))$$
Here χ^T is the well-known Serre's intersection multiplicity. In particular, if $\dim M + \dim N \leq \dim R = \dim T - 1$, then $\theta^R(M,N) = 0$ (note that vanishing for χ^T is proved for all regular local rings; see [24, 13.1]*)*

(2) *$\theta^R(M,N)$ is bi-additive on short exact sequences, assuming it is defined on all pairs. In particular, if M is locally of finite projective dimension on U_R, then the rule: $[N] \mapsto \theta^R(M,N)$ induces maps $\overline{G}(R) \to \mathbb{Z}$ and $\overline{G}(R)_{\mathbb{Q}} \to \mathbb{Q}$.*

The following elementary but useful result will be used in the proof of our main Theorem:

Lemma 2.3. *(Lemma 2.3,* [7]*) Let R be a Cohen-Macaulay local ring, M,N finitely generated R-modules and $n > 1$ an integer. Consider the two conditions:*

(1) $\mathrm{Hom}(M,N)$ *satisfies Serre's condition* (S_{n+1}).
(2) $\mathrm{Ext}^i_R(M,N) = 0$ *for* $1 \leq i \leq n-1$.

If M is locally free in codimension n and N satisfies (S_n), then (1) implies (2). If N satisfies (S_{n+1}), then (2) implies (1).

Finally we shall need a refined version of the Bourbaki sequence for a module:

Theorem 2.4. *(*[15, Theorem 1.4]*) Let R be a commutative, Noetherian ring satisfying condition (S_2). Let M be a torsion-free R-module and S be a finite set of prime ideals of R. Assume that M is free of constant rank on S and the set of height at most 1 primes in R. Then there is a Bourbaki sequence $0 \to F \to M \to I \to 0$ such that $I \nsubseteq \bigcup_{P \in S} P$.*

3. Main results

Throughout this Section R will be a local hypersurface of dimension 3. All modules are finitely generated. Note that since $\operatorname{depth} R > 2$, U_R is connected, so any module which is locally free on U_R also has constant rank.

Proposition 3.1. *Let M be reflexive R-module which is locally free of constant rank on U_R. Let N be an R-module which is locally free of constant rank on the minimal primes of R and such that $c_1([N])$ is torsion in $\mathrm{CH}^1(R)$. Then $\theta^R(M,N) = 0$.*

Proof. Without loss of generality one can assume $c_1([N]) = 0$ by replacing N with a direct sum of copies of N if necessary. First we claim that in $\overline{G}(R)_{\mathbb{Q}}$, the reduced Grothendieck group with rational coefficients, we have an equality

$[N] = \sum a_i[R/P_i]$ such that each $P_i \in \operatorname{Spec} R$ has height at least 2. Since N has constant rank a we have a short exact sequence:

$$0 \to R^a \to N \to N' \to 0$$

where N' is a torsion module. Let $\mathcal{F}$ be a prime filtration of N'. Clearly $\mathcal{F}$ involves only primes of height at least 1. Let s be the formal sum of all height 1 primes in $\mathcal{F}$. Since $c_1([N'] = c_1([N]) = 0$ we have formally (see Subsection 2.2):

$$s = \sum n_j \operatorname{div}(f_j, R/q_j)$$

here the n_j are integers, each q_j is a minimal prime of R and f_j is a regular element in R/q_j and by definition:

$$\operatorname{div}(f_j, R/q_j) = \sum \ell(R/(q_j, f_j)_p)[R/p]$$

(the sum runs over all primes of height 1 in $\operatorname{Supp}(R/(q_j, f_j))$. The above formal equality shows that in $G(R)$ one has:

$$[N'] = \sum n_j[R/(q_j, f_j)] + \sum a_i[R/p_i]$$

such that all the primes p_i are of height at least 2 and the a_i are integers. But the exact sequence $0 \to R/q_j \to R/q_j \to R/(f_j, q_j) \to 0$ shows that each $[R/(q_j, f_j)] = 0$ in $G(R)$, so our claim follows.

Because of the claim above we will be done by showing that $\theta^R(M, R/P) = 0$ for each $P \in \operatorname{Spec} R$ such that height $P \geq 2$.

By Theorem 2.4 one can construct a Bourbaki sequence for M:

$$0 \to F \to M \to I \to 0$$

such that $I \subsetneq P$. Obviously $\theta^R(M, R/P) = \theta^R(I, R/P)$. But $R/I \otimes_R R/P$ has finite length, and $\dim R/I + \dim R/P \leq 3 = \dim R$. By 2.2 $\theta^R(R/I, R/P) = 0$. Since $\theta^R(I, R/P) = -\theta^R(R/I, R/P)$ we are done.

□

Proposition 3.2. *Let $M \in \operatorname{MCM}(R)$ such that M is locally free on U_R and N be any finitely generated R-module. Suppose that $\theta^R(M^*, N) = 0$. If $\operatorname{Ext}^1_R(M, N) = 0$ then M is free or $\operatorname{pd}_R N < \infty$.*

Proof. One has the following short exact sequence (see [13, 3.6] or [18], [19]):

$$\operatorname{Tor}_2^R(M_1, N) \to \operatorname{Ext}^1_R(M, R) \otimes_R N \to \operatorname{Ext}^1_R(M, N) \to \operatorname{Tor}_1^R(M_1, N) \to 0$$

Here M_1 is the cokernel of $F_1^* \to F_2^*$, where $\mathbf{F} : \cdots \to F_2 \to F_1 \to F_0 \to M \to 0$ is a minimal resolution of M. Since $\operatorname{Ext}^1_R(M, N) = 0$ it follows that $\operatorname{Tor}^1_R(M_1, N) = 0$.

Since M is MCM and R is a hypersurface we know that the minimal resolution $\mathbf{F}$ is periodic of period at most 2 (see [10] and $\operatorname{Ext}^i_R(M, R) = 0$ for $i > 0$. It follows that the dual complex F^* is also exact and periodic of period at most 2. Thus M_1 is isomorphic to the first syzygy of M^*. In particular, M_1 is maximal Cohen-Macaulay or zero. Since $\theta^R(M_1, N) = -\theta^R(M^*, N) = 0$, it now follows that $\operatorname{Tor}_i^R(M_1, N) = 0$ for all $i > 0$ (as M_1 is maximal Cohen-Macaulay, the sequence of modules $\{\operatorname{Tor}_i^R(M_1, N)\}$ is periodic of period 2 for $i > 0$). So either M_1 or N has finite projective dimension by [14, Theorem 1.9] or [22, 1.1]. But if M_1 has finite projective dimension and is non-zero, it must be free by the Auslander-Buchsbaum formula, contradicting the minimality of $\mathbf{F}$. Thus M_1 is zero and M must be free.

□

Corollary 3.3. *Let $(R,\mathfrak{m})$ be local hypersurface of dimension 3. Let N be a reflexive R-module which is locally free on U_R. Assume that $\theta^R(N^*,N)=0$. Then $\operatorname{Hom}_R(N,N)\in \operatorname{MCM}(R)$ if and only if N is free.*

Proof. The sufficient direction is trivial. Suppose that $\operatorname{Hom}_R(N,N)$ is maximal Cohen-Macaulay. Then by Lemma 2.3 $\operatorname{Ext}^1_R(N,N)=0$. We look at the MCM approximation of N as in 2.3:

$$0\to W\to M\to N\to 0$$

As the discussion in 2.3 indicates, W is free. Applying $\operatorname{Hom}_R(-,N)$ we obtain $\operatorname{Ext}^1_R(M,N)=0$. Also, applying $\operatorname{Hom}_R(-,R)$ yields:

$$0\to N^*\to M^*\to W^*\to \operatorname{Ext}^1_R(N,R)\to 0$$

since $\operatorname{Ext}^1_R(M,R)=0$ as its Matlis dual is $\mathrm{H}^2_{\mathfrak{m}}(M)=0$. Note that $\theta^R(L,-)$ is always defined if L is any of the four modules in the above exact sequence, and it is 0 when $L=W^*$ or $L=\operatorname{Ext}^1_R(N,R)$ (the latter is because $\operatorname{Ext}^1_R(N,R)$ has finite length and Proposition 2.2). So $\theta^R(M^*,N)=\theta^R(N^*,N)=0$.

Proposition 3.2 shows that either M is free or $\operatorname{pd}_R N<\infty$. Both possibilities imply that $\operatorname{pd}_R N<\infty$. As N is reflexive and $\dim R=3$, $\operatorname{pd}_R N\le 1$. But $\operatorname{Ext}^1_R(N,N)=0$, so $\operatorname{pd}_R N$ can not be 1 by Nakayama's Lemma, thus N is free. □

We have gathered enough to prove our main result:

Theorem 3.4. *Let R be local hypersurface of dimension 3. Let N be a reflexive R-module which is locally free on U_R. Furthermore, assume that the image $c_1([N])$ (of N as an element in $G(R)$) is torsion in $\operatorname{CH}^1(R)$. Then $\operatorname{Hom}_R(N,N)\in\operatorname{MCM}(R)$ if and only if N is free.*

Proof. A combination of Proposition 3.1 and Corollary 3.3 give the desired result. □

Corollary 3.5. *Let R be local hypersurface of dimension 3. Then $\operatorname{Pic} U_R$ is torsion-free.*

Proof. Let $\mathcal{E}$ represent a torsion element in $\operatorname{Pic} U_R$. By 2.1 $I=\Gamma_X(\mathcal{E})$ is a reflexive ideal which is locally free of rank 1 on U_R. By the diagram in Subsection 2.2 we know that $c_1([I])$ is torsion in $\operatorname{CH}^1(R)$. Theorem 3.4 now applies directly to give the desired result. □

Finally we note some interesting consequences of the main results above:

Theorem 3.6. *Let R be a local hypersurface with isolated singularity and $\dim R=3$. The following are equivalent:*

1. *$\theta^R(M,N)=0$ for all $M,N\in\operatorname{mod}(R)$.*
2. *R is a unique factorization domain (equivalently, $\operatorname{CH}^1(R)=\operatorname{Cl}(R)=0$).*
3. *The class group $\operatorname{Cl}(R)$ is torsion.*

Proof. First, since R is local and normal (by Serre's criterion), it is a domain (see [21, Theorem 23.8]). Assume (1). Let I be a reflexive ideal representing an element of $\operatorname{Cl}(R)$. Then $\operatorname{Hom}_R(I,I)\cong R$, and Corollary 3.3 implies I is principal, so $\operatorname{Cl}(R)=0$.

The implication (2) $\Rightarrow$ (1) follows from 3.1. The equivalence (2) $\Leftrightarrow$ (3) is implied by main Theorem 3.5.

□

Remark. If $\hat{R}$ is a hypersurface in an equicharacteristic or unramified regular local ring then the above result follows from [5, Corollary 3.5] and [6, Theorem 3.4]. We also note that the equivalence (1) $\Leftrightarrow$ (3) when $k = \mathbb{C}$ and R is graded is obtained in [23, 3.10, 6.1].

4. Open questions

In this section we discuss some open questions motivated by the results obtained previously. Clearly, the most important question is whether or not Theorem 1.3 is true when R is a local complete intersection of dimension 3. Affirmation of such a statement would immediately prove Gabber's Conjecture 1.1. Since for an R-module M, $c_1([M])$ will be torsion in $\mathrm{CH}^1(R)$ if $[M] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$, a possibly weaker but somewhat less technical version would be:

Conjecture 4.1. *Let R be local complete intersection of dimension 3. Let N be a reflexive R-module which is locally free of constant rank on U_R. Furthermore, assume that $[N] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$, the reduced Grothendieck group of R with rational coefficients. Then $\mathrm{Hom}_R(N, N)$ is a maximal Cohen-Macaulay R-module if and only if N is free.*

In view of the proof of the key Proposition 3.2 and previously known results for regular and hypersurface rings, we feel it is reasonable to make:

Conjecture 4.2. *Let R be local complete intersection (of arbitrary dimension). Let M, N be R-modules such that M is locally free of constant rank on U_R and $[N] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$. Then (M, N) is* Tor*-rigid, in the sense that for any $i > 0$, $\mathrm{Tor}_i^R(M, N) = 0$ forces $\mathrm{Tor}_j^R(M, N) = 0$ for $j \geq i$.*

By going through the proofs of 3.2 and 3.3 one can see easily that an affirmative answer to Conjecture 4.2 (in dimension 3) would imply Conjecture 4.1.

Tor-rigidity has been a subject of active investigation in commutative algebra. For more in-depth discussion and references, we refer to the introduction of [5] and the bibliography there. It is well-known that if R is regular then Tor-rigidity holds for any pair of modules by work of Auslander and Lichtenbaum [2, 20]. Furthermore, Conjecture 4.2 is known when $\hat{R}$ is a hypersurface in an equicharacteristic or unramified regular local ring, see [5, 6]. A simple unknown case is when M, N are 0-dimensional, in such situation the conditions on M and N are automatic, so the above Conjecture would just say that any pair of finitely generated R-modules of finite length over a local complete intersection is Tor-rigid. The case when one of the modules has finite length is discussed in the last section of [6], and is still unknown to the best of our knowledge.

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